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Special Functions for Engineers

Abstract. Well established *special functions* are an important tool to expand analytical methods in many engineering applications. Unfortunately, they have fallen out of fashion in engineering educational programs and are mainly replaced by direct numerical computations. But since they simplify the analysis of many practical problems, they should find a place again in engineering curricula. This claim is substantiated by examples from fluid mechanics, production planning, mechanics and vibration theory, employing the Lambert W function, the error function, elliptic and Bessel functions.

Introduction

The standard mathematical education of engineering students is usually divided into two different parts: In the first three semesters they learn basic concepts and calculation techniques from linear algebra, analysis and ordinary differential equations, and immediately apply them in mechanics, electrical engineering and vibration theory. The focus is on manual computations, using parameters to solve general problems and find optimal values. In the second part they come in touch with "real-world" problems and find that manual computations don't work anymore. This is the time for topics such as numerical mathematics, modeling and simulation and finite element computations. Now, parameter studies are done numerically, optimization often becomes a major numerical task. Usually missing in the curriculum are strategies to extend analytical methods, which can make parameter studies and optimization much simpler.

A conceptionally easy way to extend the computation capabilities is the introduction of *special functions*, which are well-known in applications and whose properties have been extensively studied [1]. This notion has no formal definition, some authors use the hypergeometric series or special classes of differential equations as unifying criterion [2]. We will stick here to the general consensus, including non-elementary functions that have proven their value in applications. Though many of them have interesting properties, when extended to the complex numbers, we will concentrate on real functions here.

In the following, we will use four different methods to define new function on $\mathbb{R}$: as inverse functions, antiderivatives, definite integrals with parameters or solutions of ordinary differential equations. Of course there are a lot of other possibilities, e.g. power series or integral transforms. In each case, we will shortly present an application problem, then introduce the special function needed, study some of its properties, and finally show, how to solve the original problem using them.

Inverse Functions

The first example is a standard problem in basic fluid mechanics lectures [3]: Given a horizontal pipe of length $l = 2$ km, diameter $d = 0.5$ m and surface roughness $k = 0.1$ mm that is used to transport a volume flow $\dot{V} = 1200\,\mathrm{m}^3/\mathrm{h}$ of hot water of 85 °C. Compute the pressure loss Δp in the pipe. Following the standard procedure leads to an apparently simple equation that can't be solved analytically.

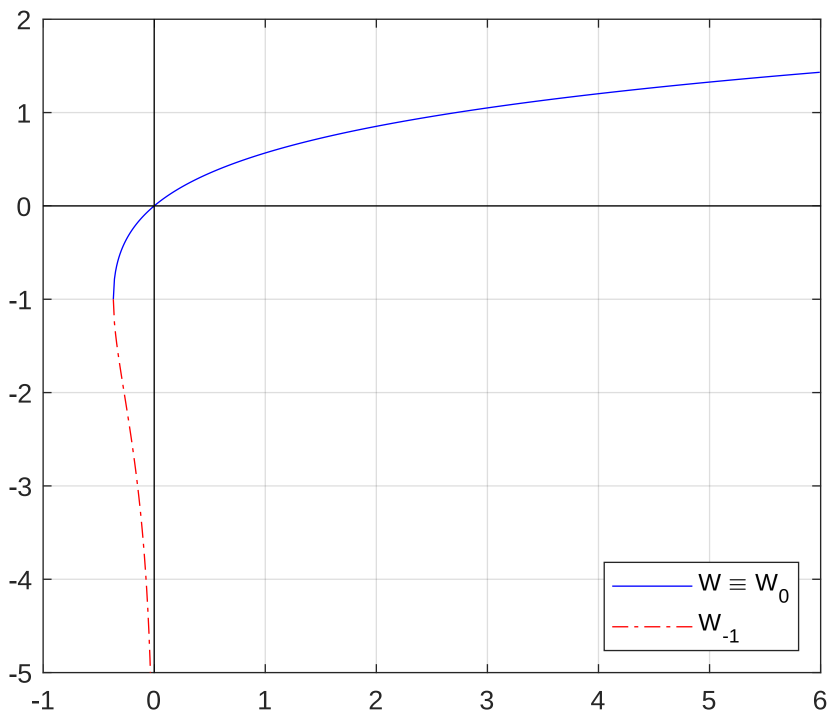

Figure 1: Lambert W function.

The *Lambert W function* is the upper branch of the inverse of $f(x) = x\,e^x$ for $x > -1/e$ (cf. Fig. 1). It is sometimes denoted as W_0, while the lower branch is named W_{-1}.

Amongst its many useful properties are (for suitable x, a)

$$\begin{aligned}
x\, e^x = a \quad &\Rightarrow \quad x = W(a) \\
x \ln x = a \quad &\Rightarrow \quad x = e^{W(a)} \\
x^x = a \quad &\Rightarrow \quad x = \frac{\ln a}{W(\ln a)} \\
W'(x) &= \frac{W(x)}{x(1 + W(x))} \\
\int W(x)\, dx &= x \left(W(x) - 1 + \frac{1}{W(x)} \right) + C
\end{aligned}$$

They are summarized in [4], together with many of its applications. In Matlab it is defined directly as `lambertw(x)`, the negative branch as `lambertw(-1, x)`.

Its applications span a lot of different areas, e. g. explicit solutions of some quantenmechanical systems, the kinetics of enzyme-catalysed reactions, crystal growth, solutions of the Einstein vacuum equations or the SIR equations of epidemiology. [5] even argues that due to its rich mathematical structure and abundant applications, it should be included in the set of elementary functions and taught at secondary or tertiary school levels.

Using the W function, the solution of the introductory example proceeds as follows: First one gets the temperature-dependent values of the density ρ and the kinematic viscosity ν from water tables and computes the dimensionless quantities

$$\begin{aligned}
\mathrm{Re} &= \frac{\overline{w}\, d}{\nu} = \frac{4\, \dot{V}}{\pi\, d\, \nu} \\
k_{rel} &= \frac{k}{d}
\end{aligned}$$

Next, one computes the Darcy friction factor λ by solving the Coleman equation

$$\frac{1}{\sqrt{\lambda}} = -\frac{2}{\ln 10} \ln \left(\frac{2.51}{\mathrm{Re}} \frac{1}{\sqrt{\lambda}} + \frac{k_{rel}}{3.7} \right).$$

To this end one introduces $x = 1/\sqrt{\lambda}$ and some obvious abbreviations to write it as

$$x = -c \ln(ax + b).$$

A bit of simple algebra leads to

$$\left(\frac{x}{c} + \frac{b}{ac}\right) e^{\frac{x}{c} + \frac{b}{ac}} = \frac{e^{\frac{b}{ac}}}{ac}$$

with the solution

$$x = c\, W\left(\frac{e^{\frac{b}{ac}}}{ac}\right) - \frac{b}{a}$$

leading to the numerical value

$$\lambda = 0.01415.$$

Finally the pressure loss is computed as

$$\Delta p = \lambda \frac{l}{d} \frac{\rho}{2} \overline{w}^2 = 0.7900 \,\text{bar}\,.$$

Antiderivatives

The second example is concerned with first steps in production process planning: In a manufacturing process, resistors of $R = 47$ kΩ must be produced. They should belong to the norm series E12, i. e. the resistance R may not deviate from the specified value by more than 10%. The values actually produced are normally distributed with $E(X) = 47$ kΩ and $\sigma(X) = 3$ kΩ. Calculate the percentage of resistors produced that are in the permissable range. The mathematical problem is, of course, to compute the antiderivative of $f(x) = e^{-x^2}$.

The *error function* is defined as

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

(cf. Fig. 2). Since a basic course in stochastics is now standard in engineering curricula, this function is actually known to the students, together

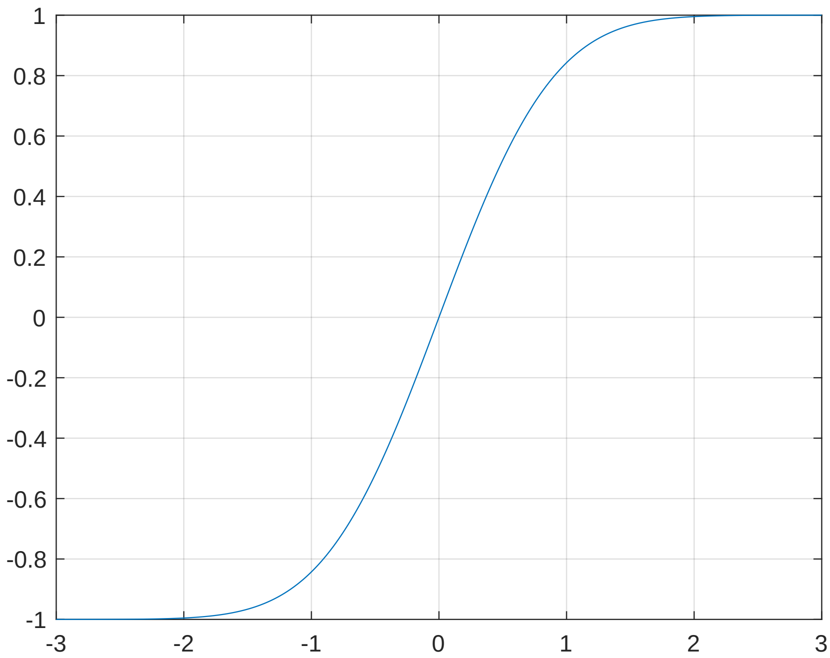

Figure 2: Error function.

with its basic properties

$$\begin{aligned} \operatorname{erf}(-x) &= -\operatorname{erf}(x) \\ \operatorname{erf}(0) &= 0 \\ \lim_{x\to\infty} \operatorname{erf}(x) &= 1 \end{aligned}$$

In Matlab it is given as `erf(x)`, and its complement by `erfc(x) = 1 - erf(x)`.

Related functions are the Fresnel integrals

$$\begin{aligned} S(x) &:= \int_0^x \sin\frac{\pi}{2}t^2\,dt \\ C(x) &:= \int_0^x \cos\frac{\pi}{2}t^2\,dt, \end{aligned}$$

which can be reduced to the error function of complex arguments.

As a consequence of the central limit theorem, the error function appears in lots of statistics applications. A variant is the Maxwell distribution function of molecule velocities in an ideal gas. Furthermore, it is important for the computation of heat conduction phenomena, since it is the fundamental solution of the heat equation. The Fresnel integrals describe the scattering of light around obstacles in the near-field region and – a bit surprisingly – optimal curves for motorway exits [6].

Exercise 2 is a standard example in stochastics: The distribution of the resistance values is given by

$$X \sim N(47, 9),$$

the percentage is

$$p = P(0.9 \cdot 47 \leq X \leq 1.1 \cdot 47).$$

Normalizing with

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

then leads to

$$p = P\Big(-\frac{4.7}{3} \leq Z \leq \frac{4.7}{3}\Big) = \operatorname{erf}\Big(\frac{4.7}{3\sqrt{2}}\Big) = 88.28\%.$$

Definite Integrals

The third exercise is a standard mechanics example: Given a mathematical pendulum of length $l = 1$ m, calculate the oscillation period for the initial values

$$\varphi_0 = 10°/90°/175°, \ \dot{\varphi}_0 = 0.$$

The difficult mathematical problem here is the computation of a definite integral with a parameter.

The *complete elliptic integral of the first kind* is defined as

$$K(m) := \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - m \sin^2 \vartheta}}, \quad m \in [0, 1)$$

(cf. Fig. 3). It has a lot of interesting properties, amongst them

$$K(0) = \frac{\pi}{2}, \quad K(1) = \infty$$

$$K(m) = \frac{\pi}{2}\left(1 + \left(\frac{1}{2}\right)^2 m + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 m^2 + \dots\right)$$

$$K(m) = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-mt^2)}}$$

In Matlab it can be calculated with `ellipke(m)`.

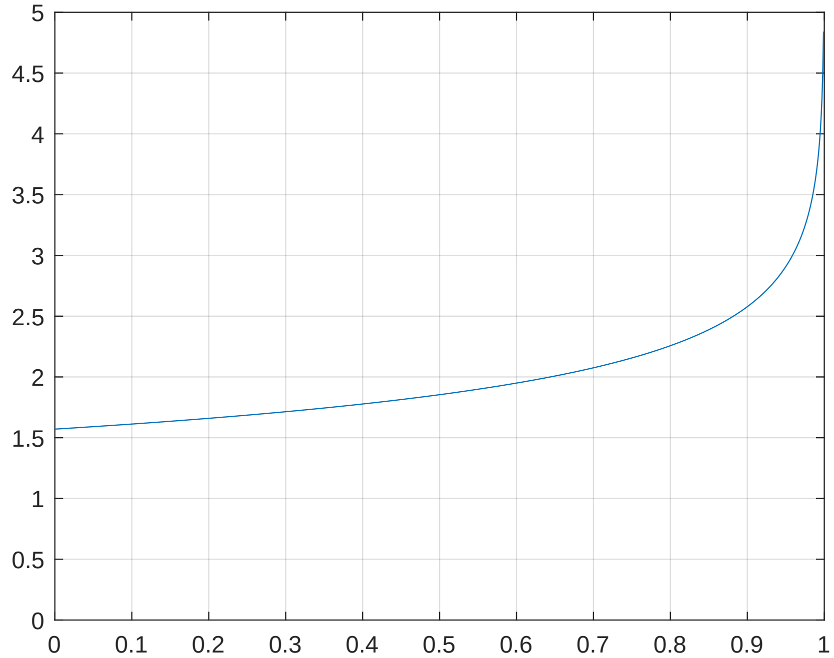

Figure 3: Complete elliptic integral of the first kind.

K belongs to the important family of elliptic integrals

$$\int R(x, \sqrt{P(x)})\, dx,$$

where $P(x)$ is a polynomial of degree 3 or 4 (with 3 or 4 different roots to exclude "trivial" cases) and $R(x, y)$ is a non-trivial rational function of its two arguments. An important special case is the *incomplete elliptic integral of the first kind*

$$F(u, m) = \int_0^u \frac{1}{\sqrt{(1-t^2)(1-mt^2)}}\, dt$$

with the connection

$$K(m) = F(1, m).$$

The inverse of F as function of u is called *Jacobi elliptic function sn*, it can be considered as a generalization of a harmonic function using a special nonlinear restoring force [7].

The elliptic integrals and elliptic functions play a prominent role in mathematics [8]: One of their first appearances was in the computation of the arc length of the ellipse (hence their name). In complex function theory, elliptic functions are defined as meromorphic, doubly periodic functions, with a deep connection to the elliptic integrals. They are the starting point of modern developments from elliptic curves to modular forms up to the proof of Fermat's theorem. They even provide an explicit solution of the quintic equation. They appear in many applications, such as the trajectory of the mathematical pendulum, the form of a skipping rope, the description of soliton waves or even in cryptography (ok, this needs a few further steps from here).

The solution of exercise 3 starts with the equation of motion

$$\ddot{\varphi} + \omega^2 \sin\varphi = 0$$

with

$$\omega = \sqrt{\frac{g}{l}}.$$

Multiplying with $\dot{\varphi}$ and integrating (or simply by energy conservation) one gets

$$\dot{\varphi}^2 = 2\omega^2(\cos\varphi - \cos\varphi_0)$$

Integration from 0 to $T/4$ leads to

$$T = \frac{2\sqrt{2}}{\omega}\int_0^{\varphi_0} \frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}}$$

The – not so obvious – substitution

$$\sin u = \frac{\sin\frac{\varphi}{2}}{\sin\frac{\varphi_0}{2}}$$

leads to

$$T = \frac{4}{\omega} \int_0^{\pi/2} \frac{du}{\sqrt{1 - \sin^2 \frac{\varphi_0}{2} \sin^2 u}}.$$

Introducing the oscillation period T_0 for very small initial φ_0

$$T_0 := \frac{2\pi}{\omega},$$

one finally gets

$$T = \frac{2T_0}{\pi} K\left(\sin^2 \frac{\varphi_0}{2}\right).$$

Numerically the results for the different initial conditions are:

φ_0	T	$\frac{\lvert T-T_0\rvert}{T_0}$
10°	2.0521	0.0229
90°	2.6640	0.3280
175°	6.2141	2.0977

Obviously, the usual approximation $\sin\varphi \approx \varphi$ for small initial angles, leading to $T = T_0$, is completely wrong for values approaching 180°: The pole of $K(m)$ at $m = 1$ corresponds to the – unstable – equilibrium point at $\varphi_0 = \pi$.

Differential Equations

The final example is a standard problem from vibration theory: Given a circular membrane with wave speed c that is fixed at radius r_0, calculate the frequencies of the first 10 vibration modes. c can be computed from membrane properties such as elasticity and density and the tension forces applied to the membrane. Solving the wave equation for a circular geometry leads to an ordinary differential equation with solutions that cannot be expressed with elementary functions.

The *Bessel equation* is defined by

$$x^2 f'' + x f' + (x^2 - m^2) f = 0$$

for $m = 0, 1, 2, \ldots$ and positive x. For given m, the equation has two linearly independent solutions J_m and Y_m, the *Bessel functions* of the first and second kind, where $J_m(0)$ is finite, whereas Y_m has a pole at 0 (cf. Fig. 4). They have infinitely many zeros, which will be denoted by $j_{m,n}$ and $y_{m,n}$ for $n = 1, 2, \ldots$.

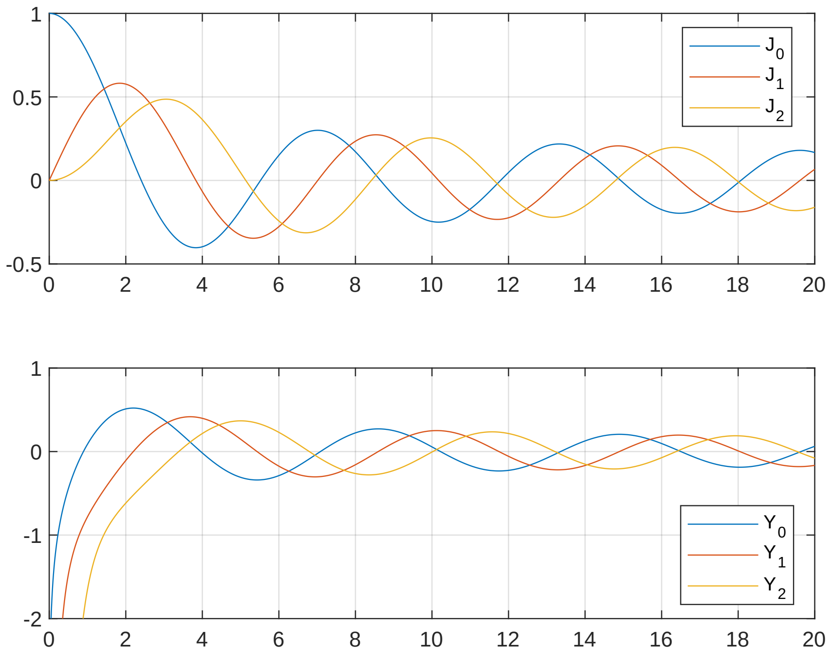

Figure 4: Bessel functions.

The Bessel functions satisfy the recurrence relations

$$\frac{2m}{x} J_m(x) = J_{m-1}(x) + J_{m+1}(x)$$
$$2J'_m(x) = J_{m-1}(x) - J_{m+1}(x)$$

and the orthogonality relation [9]

$$\int_0^\infty J_m(x) J_n(x) \, \frac{dx}{x} = \frac{2}{\pi} \frac{\sin\left(\frac{\pi}{2}(m-n)\right)}{m^2 - n^2}.$$

For large x they are approximately harmonic

$$J_m(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right), \quad x \gg 1.$$

In Matlab values of the Bessel functions can be calculated with `besselj(m,x)` and `bessely(m,x)`. Their zeros can be computed with `fzero`, but a much faster and more reliable function `besselzero` is available at the MATLAB Central File Exchange [10].

Closely related functions are the Hankel functions, which are complex linear combinations of J_m and Y_m. Furthermore, solutions of the Bessel equations are studied for arbitrary (non-integer) values of m. Especially useful in applications are the solutions with half-integer m, the *spherical Bessel functions*.

The Bessel functions appear in all kinds of physical applications with cylindrical symmetry, e. g. electromagnetic waves in a waveguide, heat conduction, wave functions in quantum mechanics or diffraction through an aperture. In corresponding situations with spherical symmetry, the spherical Bessel functions are similarly useful.

To solve exercise 4, one starts by writing the two-dimensional wave equation using polar coordinates:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi} = \frac{1}{c^2}u_{tt}.$$

For the computation of eigen modes, one assumes a harmonic time behaviour and a separation of space variables:

$$\begin{aligned} u(r,\varphi,t) &= f(r)g(\varphi)\cos(\omega t) \\ \Rightarrow \qquad g(\varphi) &= A\cos(m\varphi + \varphi_0), \quad m = 0, 1, 2, \ldots \end{aligned}$$

Introducing $k := \omega/c$ and changing the variable from r to $z := k\,r$, one gets the Bessel equation for $f(z)$. Certainly, the vibration must be finite at $r = 0$, therefore the J_m are the only solutions. Since the membrane is fixed at the circle $r = r_0$, we have

$$J_m(k\,r_0) = 0,$$

giving eigen frequencies at

$$\omega = \frac{c}{r_0} j_{m,n}.$$

A table of zeros $j_{m,n}$ for $m = 0 \ldots 5$ and $n = 1 \ldots 4$ provides enough values to find the ten lowest modes and corresponding frequencies:

n m	1	2	3	4
0	*2.40	*5.52	*8.65	11.79
1	*3.83	*7.02	10.17	13.32
2	*5.14	*8.42	11.62	14.80
3	*6.38	9.76	13.02	16.22
4	*7.59	11.06	14.37	17.62
5	*8.77	12.34	15.70	18.98

Conclusions

Special functions play an important role in many applications, especially, but not confined to electrical and mechanical engineering, physics and statistics. Consequently, their definitions and properties should be included in engineering curricula. A standard mathematics course in analysis and differential equations seems to be an appropriate place to include this topic. But since the time is already too short to cover the basic material, this seems unfeasible in practice. In a non-systematic fashion, special functions could be included as examples or in homework exercises. But it would probably be better to integrate them into application courses as soon as they are needed.

Of course, special functions are an interesting topic for mathematicians as well: They provide lots of important examples, allow to extend tools from analysis and often are starting points for complete new areas. The general tendency to teach mathematics by starting with abstract notions often leaves students without a proper understanding of the motivation behind the definitions. Topics like special functions can provide a basis for later abstraction and a feeling for the many interconnections between seemingly unrelated areas of mathematics.

Unfortunately, special functions have fallen out of fashion in teaching engineering students and are often replaced by numerical methods. This can have severe consequences, e. g. when optimization with many parameters is done directly using complex and computationally intense numerical methods, where applying analytical methods could lead to much faster and more reliable computations. Hopefully, the explicit examples provided here have shown that special functions deserve a renaissance, not only for practical purposes, but in the spirit of Hammings famous motto:

The purpose of computing is insight, not numbers. [11]

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